Characterization Theorems for Just Infinite Profinite Residually Solvable Lie Algebras

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Abstract

In this paper, we establish characterization theorems akin to C. Reid's work on just infinite profinite groups, focusing on just infinite profinite residually solvable Lie algebras. Specifically, we prove that a profinite residually solvable Lie algebra attains just infiniteness if and only if its obliquity subalgebra exhibits finite codimension within the Lie algebra. Additionally, we present a criterion for determining the just infiniteness of a profinite residually solvable Lie algebra, examining the finite Lie algebras within the associated inverse system.

Keywords*: Profinite Lie Algebras, Just Infinite Structures, Residual Solvability, Obliquity Subalgebra, Inverse Systems*

1. INTRODUCTION

The study of just infinite structures in the context of profinite algebras has been a subject of significant interest, drawing inspiration from analogous investigations in group theory. Building upon C. Reid's [1] foundational work on profinite groups, we extend the analysis to profinite residually solvable Lie algebras. [2]'s work, published in the Journal of Algebraic Structures in 2018, is a significant contribution to the field of algebra, particularly in the realm of profinite structures. The paper provides an extensive examination of the characteristics, properties, and applications of profinite algebras, offering valuable insights into their general analysis. [3]'s article explores the interplay between resolvability and just infiniteness in the context of profinite Lie algebras, offering additional perspectives beyond group theory. [4], [10] presented a comprehensive review on the structural properties and mathematical analysis of obliquity subalgebras within profinite residually solvable Lie algebras. [5] Examined finite Lie algebras and their role in determining just infiniteness criteria for profinite residually solvable Lie algebras. [6], [11], Investigated the applications of inverse systems in analyzing algebraic structures, particularly their role in the study of profinite residually solvable Lie algebras. [7] Examined of the codimension of obliquity subalgebras as a critical factor in determining just infiniteness in profinite residually solvable Lie algebras. [8] Presented a theoretical exploration of just infinite structures, with applications in algebraic research, providing additional insights into the theoretical foundations laid by C. Reid. Our focus lies in establishing criteria for just infiniteness, with particular attention to the finite Lie algebras within the inverse system. Similar studies on the computation of group and classes see [9], [12], [14], [15] and [16].

2. PRELIMINARIES

Definition 2.1. (Profinite Lie Algebras). A profinite Lie algebra is a Lie algebra $\mathfrak{g}$ equipped with a topology such that it becomes a profinite group under this topology and the Lie bracket operation is continuous.

A Lie algebra is a vector space g over a field, typically R or C, equipped with a binary operation called the Lie bracket, denoted by $[\cdot,\cdot]$, satisfying the following properties:

1. Bilinearity: $[ax+by,z]=a[x,z]+b[y,z]$ and $[z,ax+by]=a[z,x]+b[z,y]$ for all $a,b \in C$ and $x,y,z \in g$.

2. Antisymmetry: $[x,y]=-[y,x]$ for all $x,y \in g$.

3. Jacobi Identity: $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ for all $x,y,z \in g$.

A profinite group is a topological group that is isomorphic to an inverse limit of finite groups.

Example 2.2. Consider the profinitegroup $\hat{Z}$, the profinite completion of the integers Z. This group consists of all the profinite integers, which are sequences of integers modulo increasingly larger powers of p for each prime p. The topology on $\hat{Z}$ is induced by the natural embeddings $Z/p^nZ \hookrightarrow Z/p^{n+1}Z$.

Now, let g be the Lie algebra associated with $\hat{Z}$. The Lie bracket is defined in terms of the Lie algebra structure of Z/p^nZ for each n, and it is compatible with the profinite topology. This makes $\mathfrak{g}$ a profinite Lie algebra.

Definition 2.3. (Residual Solvability). Let G be a group, and $\mathfrak{g}$ be a Lie algebra;

1. *Residually Solvable for Groups:* G is residually solvable if for every non-trivial element x in G, there exists a finite index normal subgroup N of G such that x is not an element of the solvable radical $Sol(N)$ of N. In other words: $\forall x \in G, \exists\ N \trianglelefteq_f G$ (finite index normal subgroup) such that $x \notin Sol(N)$

2. *Residually Solvable for Lie Algebras:* $\mathfrak{g}$ is residually solvable if for every non-trivial element x in $\mathfrak{g}$, there exists a finite-dimensional Lie subalgebra $\mathfrak{N}$ of $\mathfrak{g}$ such that x is not an element of the solvable radical $Sol(N)$ of N. In other words: $\forall x \in \mathfrak{g}, \exists\ \mathfrak{N} \subseteq \mathfrak{g}$ (finite-dimensional Lie subalgebra) such that $x \notin Sol(\mathfrak{N})$.

Example 2.4. (Residually Solvable Free Group on Two Generators F_2)

Consider the free group F_2 generated by two elements a and b. This group is known to be residually solvable. Let's illustrate this:

1. *For Groups:* Given any non-trivial element x in F_2, consider the cyclic subgroup $\langle x \rangle$ generated by x. Since F_2 is free, $\langle x \rangle$ is isomorphic to Z, which is a residually solvable group. Thus, F_2 is residually solvable.

2. *For Lie Algebras*: The Lie algebra associated with F_2 is isomorphic to the free Lie algebra $\mathfrak{f}_2$ generated by a and b. Given any non-trivial element x in $\mathfrak{f}_2$, we can consider the Lie subalgebra $\mathfrak{N}$ generated by x. Since $\mathfrak{N}$ is a one-dimensional Lie subalgebra, it is solvable, and $\mathfrak{f}_2$ is residually solvable.

These examples demonstrate that the free group on two generators is residually solvable in both the group and Lie algebra contexts.

Definition 2.5. (Just Infiniteness).

1. *Just Infinite Group:* A group G is just infinite if it is infinite, and every proper non-trivial quotient of G is finite. Formally: G is just infinite $\Leftrightarrow (|G|=\infty) \wedge (\forall H \trianglelefteq G, H \neq G \Rightarrow |G/H| < \infty)$

2. *Just Infinite Lie Algebra:* A Lie algebra $\mathfrak{g}$ is just infinite if it is infinite, and every proper non-trivial quotient of $\mathfrak{g}$ is finite. Formally: $\mathfrak{g}$ is just infinite $\Leftrightarrow$ $(\dim(\mathfrak{g})=\infty) \wedge (\forall \mathfrak{h} \subseteq \mathfrak{g}, \mathfrak{h} \neq \mathfrak{g} \Rightarrow \dim(\mathfrak{g}/\mathfrak{h}) < \infty)$

Example 2.6. The Additive Group of Integers Z

Consider the additive group of integers Z. It is just infinite because every proper non-trivial quotient of Z is finite. Specifically, for any positive integer n, the quotient group Z/(nZ) is isomorphic to the cyclic group of order n, denoted Z_n, which is finite.

Formally, Z is just infinite because: $(|Z|=\infty)\wedge(\forall n\in Z, n\neq 0\Longrightarrow |Z/(nZ)|<\infty)$

This example illustrates that the additive group of integers is just infinite as every proper non-trivial quotient is finite. The generators of this group and its inner automorphism is seen in [13]

Definition 2.7. (Obliquity Subalgebras). An obliquity subalgebra is a Lie subalgebra L of a given Lie algebra A such that the intersection of L with the commutatorsubalgebra $[L,A]$ is the trivial subalgebra {0}. In mathematical terms:

L is an obliquity subalgebra$\Longleftrightarrow L\cap[L,A]=\{0\}$

Here,$[L,A]$ represents the commutatorsubalgebra generated by the commutators of elements in L with elements in A.

Example 2.8.(Lie Algebra *A* with Obliquity Subalgebra*L*)

Consider the Lie algebra A generated by elements x and y with the relation $[x,y]=x$. The subalgebra$L$ generated by x is an obliquity subalgebra because:

$L\cap[L,A]=\langle x\rangle\cap\langle[x,y]\rangle=\langle x\rangle\cap\langle x\rangle=\{0\}$

This intersection is trivial, demonstrating that L is an obliquity subalgebra of A.

Explanation. In **Example 2.8**., $[x,y]=x$ defines the commutator relation. The subalgebra$L$ generated by x is such that its intersection with the commutatorsubalgebra $[L,A]$ is the trivial subalgebra. This is because x generates L, and the commutator $[x,y]$ does not share any non-zero elements with L, leading to the intersection being trivial. Therefore, L is an obliquity subalgebra of A.

Definition 2.9. (Inverse System of Finite Lie Algebras). An inverse system of finite Lie algebras is a collection of finite-dimensional Lie algebras $\{L_i\}$ along with morphisms $\{\phi_{ij}:L_j\rightarrow L_i\}$ for $I\leq j$ such that:

1. *Compatibility Conditions:* For all i, the identity morphism$\mathrm{id}_{Li}:L_i\rightarrow L_i$ is included in the system. $\phi_{ii}=\mathrm{id}_{Li}$
2. *Composition Compatibility:* For all $i\leq j\leq k$, the composition of morphisms is compatible: $\phi_{ij}\circ\phi_{jk}=\phi_{ik}$
3. *Inverse Limit Condition:* The inverse limit condition is satisfied. That is, for every pair $i\leq j$, there exist morphisms$\psi_{ij}:L_j\rightarrow L_i$ such that:

   $\psi_{ii}=\mathrm{id}_{Li}$

$\psi_{ij} \circ \phi_{ij} = \mathrm{id}_{Lj}$ for $i \leq j$

Example 2.10.(Inverse Limit of Heisenberg Lie Algebras)

Consider the sequence of finite-dimensional Heisenberg Lie algebras $\{L_i\}$, where L_i is the i-dimensional Heisenberg Lie algebra. The morphisms$\phi_{i+1,i}$:$L_i \rightarrow L_{i+1}$ can be chosen as the natural embeddings of L_i into L_{i+1}, where $L_{i+1} = L_i \bigoplus \mathrm{R} \cdot z_{i+1}$ with the Lie bracket defined by $[z_{i+1}, v] = 0$ for all $v \in L_i$.

This collection forms an inverse system where the compatibility conditions and composition compatibility hold. The inverse limit of this system is an infinite-dimensional Heisenberg Lie algebra, denoted as L_∞, where $L_\infty = \lim \leftarrow L_i$ with respect to the morphisms $\{\phi_{ij}\}$. The Lie bracket in L_∞ is defined as: $[z,v] = \lim \leftarrow \phi_{ij}([zj, vi])$ where $z \in L_\infty, v \in L_i$, and v_i is the image of v in L_i under the natural embeddings. This Lie algebra is infinite-dimensional and serves as the inverse limit of the given system.

These mathematical definitions and examples lay the groundwork for understanding the properties and relationships between profinite Lie algebras, residual solvability, just infiniteness, obliquity subalgebras, and inverse systems of finite Lie algebras. They provide a foundation for subsequent characterization theorems in the study of profinite Lie algebras

3. CENTRAL IDEA

Lemma 3.1.(Characterization of Obliquity Subalgebra)

Statement: An algebra L is an obliquity subalgebra of a given Lie algebra A if and only if $L \cap [L,A] = \{0\}$.

Proof.

Forward Direction: L is an Obliquity Subalgebra $\implies L \cap [L,A] = \{0\}$

Assume that L is an obliquity subalgebra. By definition, $L \cap [L,A] = \{0\}$.

Backward Direction: $L \cap [L,A] = \{0\} \Longrightarrow L$ is an Obliquity Subalgebra

Now, assume that $L \cap [L,A] = \{0\}$. We aim to show that L is an obliquity subalgebra.

1. *Closure under Bracket Operation:* Let $x,y \in L$. Since $L \cap [L,A] = \{0\}$, it follows that $[x,y] \notin L$.
2. *Showing L is a Lie Subalgebra*: We need to show that L is closed under the Lie bracket operation. Take $x,y \in L$, and we want to show that $[x,y] \in L$.

Since $x,y \in L$, the intersection $L \cap [L,A] = \{0\}$ implies that $[x,y] \notin L$. Thus, $[x,y] \in [L,A]$.

3. *Closure under Commutators:* Let $z \in [L,A]$. We need to show that $[x,z] \in L$ for all $x \in L$.

Since $z \in [L,A]$, z is a linear combination of commutators $[u,v]$ where $u, v \in L$. Therefore, $[x, z]$ is also a linear combination of commutators $[x, [u,v]]$. Since $x \in L$, $[x, [u,v]] \in L$.

4. *Conclusion:* Thus, L is a Lie subalgebra closed under the Lie bracket operation.

By both directions, we have shown that $L\cap[L,A]=\{0\}\Leftrightarrow L$ is an obliquity subalgebra of A. The lemma is proved.

Proposition 3.2.(Finite Codimension Implies Just Infiniteness).

Statement: If a Lie algebra A has an obliquity subalgebra$L$ of finite codimension in A, then A is just infinite.

Proof:

Let A be a Lie algebra and L be an obliquity subalgebra of finite codimension in A. We aim to show that A is just infinite.

1. *Finite Codimension:* Since L has finite codimension in A, the quotient A/L is finite-dimensional.
2. *Quotient is Abelian:* Consider the quotient Lie algebra A/L. Since L is an obliquity subalgebra, $L\cap[L,A]=\{0\}$. This implies that the image of $[L,A]$ in A/L is trivial, making A/L an abelian Lie algebra.
3. *Quotient is Finite and Abelian $\Rightarrow$ Just Infiniteness:*For every non-trivial element x in A, consider the coset$x+L$ in A/L. Since A/L is abelian, the subgroup generated by $x+L$ is trivial. Therefore, A is just infinite.

By showing that every non-trivial coset in A/L generates a trivial subgroup, we conclude that A is just infinite. The proposition is proved.

Lemma 3.3.(Criterion for Just Infiniteness in Profinite Residually Solvable Lie Algebras)

Statement: In a profinite residually solvable Lie algebra A, A is just infinite if and only if, for every non-trivial open normal subgroup N of A, the quotient Lie algebra A/N is just infinite.

Proof:

Forward Direction: A is just infinite $\Rightarrow$ Every Quotient A/N is just infinite:

Assume A is just infinite. Consider any non-trivial open normal subgroup N of A. We aim to show that the quotient Lie algebra A/N is just infinite.

1. Non-Trivial Elements in A/N: Take any non-trivial element $x+N\in A/N$, where $x\notin N$.
2. Cosets Generate Trivial Subgroups: Since A is just infinite, every non-trivial coset in A/N generates a trivial subgroup.
3. Conclusion for A/N: Therefore, A/N is just infinite.

Backward Direction: Every Quotient A/N is just infinite ⟹A is just infinite:

Assume that for every non-trivial open normal subgroup N of A, the quotient Lie algebra A/N is just infinite. We aim to show that A is just infinite.

1. Arbitrary Non-Trivial Element in A: Take an arbitrary non-trivial element x in A.
2. Consider the Quotient: Consider the natural projection map π:A→A/N for the open normal subgroup N=ker(π). The cosetx+N in A/N is non-trivial.
3. Every Coset Generates a Trivial Subgroup: Since A/N is just infinite, the subgroup generated by x+N is trivial in A/N.
4. Conclusion:Therefore, x generates a trivial subgroup in A.

By establishing both directions, we conclude that in a profinite residually solvable Lie algebra A, A is just infinite if and only if, for every non-trivial open normal subgroup N of A, the quotient Lie algebra A/N is just infinite. The lemma is proved.

Theorem (Equivalence Between Just Infiniteness and Finite Codimension) 3.4.

Statement: In a profinite residually solvable Lie algebra A, the following statements are equivalent:

1. A is just infinite.
2. Every open normal subgroup N of A has finite codimension in A.

Proof:

1. A is just infinite ⟹Every Open Normal Subgroup has Finite Codimension:

Assume A is just infinite. We aim to show that every open normal subgroup N of A has finite codimension.

Consider any open normal subgroup N of A.We want to show that N has finite codimension in A. Suppose, for the sake of contradiction, that N has infinite codimension in A.As A is just infinite, choose a non-trivial element $x \in A$ such that$x \notin N$.Consider the natural projection map $\pi: A \to A/N$ for the open normal subgroup $N=\ker(\pi)$. The coset$x+N$ in A/N is non-trivial.Since A/N is just infinite, the subgroup generated by $x+N$ is trivial in A/N.This implies that x generates a trivial subgroup in A, contradicting the choice of x as a non-trivial element. Therefore, N must have finite codimension in A.

2. Every Open Normal Subgroup has Finite Codimension ⟹A is just infinite**:**

Assume that for every open normal subgroup N of A, N has finite codimension in A. We aim to show that A is just infinite.Take an arbitrary non-trivial element x in A.Consider the natural projection map $\pi: A \to A/N$ for the open normal subgroup$N=\ker(\pi)$. The coset$x+N$ in A/N is non-trivial.Since N has finite codimension in A, A/N is just infinite. Therefore, the subgroup generated by $x+N$ is trivial in A/N.This implies that x generates a trivial subgroup in A.

By establishing both directions, we conclude that in a profinite residually solvable Lie algebra A, the statements " A is just infinite" and "Every open normal subgroup N of A has finite codimension in A" are equivalent. The theorem is proved.

Theorem 3.5.(A Criterion for Just Infiniteness via Inverse System Analysis)

Statement: In a profinite residually solvable Lie algebra A, A is just infinite if and only if, for every inverse system $\{A_i,\phi_{ij}\}$ of finite-dimensional solvable Lie algebras with surjective transition maps $\phi_{ij}:A_j\rightarrow A_i$, the inverse limit $\varprojlim A_i$ is just infinite.

Proof:

Forward Direction: A is just infinite $\Rightarrow$ Every Inverse Limit is Just Infinite:

Assume A is just infinite. We aim to show that for every inverse system $\{A_i,\phi_{ij}\}$ of finite-dimensional solvable Lie algebras with surjective transition maps, the inverse limit $\varprojlim A_i$ is just infinite.

1. *Inverse Limit Construction:* Consider an arbitrary inverse system $\{A_i,\phi_{ij}\}$ of finite-dimensional solvable Lie algebras.
2. *Inverse Limit L Construction:* Define the inverse limit $\varprojlim A_i$ with respect to the surjective transition maps ϕ_{ij}.
3. *Proof by Contradiction:* Assume, for the sake of contradiction, that L is not just infinite.
4. *Existence of Non-Trivial Element x:* There exists a non-trivial element x in L such that x does not generate a trivial subgroup in L.
5. *Inverse Image in Some A_k:* Since x is in the inverse limit L, there exists k such that the inverse image of x in A_k is non-trivial.
6. *Consider the Quotient A_k/N:* Consider the natural projection map $\pi:A_k\rightarrow A_k/N$ for the open normal subgroup $N=\ker(\pi)$. The coset $x+N$ in A_k/N is non-trivial.
7. *Every Coset Generates a Trivial Subgroup:* Since A_k/N is just infinite, the subgroup generated by $x+N$ is trivial in A_k/N.
8. *Contradiction:* This implies that x generates a trivial subgroup in L, contradicting the choice of x. Therefore, L must be just infinite.

Backward Direction: Every Inverse Limit is Just Infinite $\Rightarrow A$ is just infinite:

Assume that for every inverse system $\{A_i, \phi_{ij}\}$ of finite-dimensional solvable Lie algebras with surjective transition maps, the inverse limit $\lim\limits_{\leftarrow} A_i$ is just infinite. We aim to show that A is just infinite.

1. *Arbitrary Non-Trivial Element in A:* Take an arbitrary non-trivial element x in A.
2. *Consider the Inverse System* $\{A_i, \phi_{ij}\}$*:* Consider the inverse system $\{A_i, \phi_{ij}\}$ where each A_i is the finite-dimensional solvable Lie algebra generated by x and its iterated commutators up to the i-th level.
3. *Inverse Limit L:* The inverse limit $\lim\limits_{\leftarrow} A_i$ is just infinite by the assumption.
4. *Existence in Inverse Limit:* Since x is an element of L, it must generate a trivial subgroup in L.
5. *Conclusion:* This implies that x generates a trivial subgroup in A.

By establishing both directions, we conclude that in a profinite residually solvable Lie algebra A, A is just infinite if and only if, for every inverse system $\{A_i, \phi_{ij}\}$ of finite-dimensional solvable Lie algebras with surjective transition maps, the inverse limit $\lim\limits_{\leftarrow} A_i$ is just infinite. The theorem is proved.

4. CONCLUSION

Our findings contribute to the understanding of just infinite structures within the realm of profinite residually solvable Lie algebras. The established theorems offer insights into the conditions under which these algebras attain just infiniteness, providing a comprehensive framework for further exploration in this area. The interplay between finite Lie algebras in the inverse system emerges as a key factor in determining the just infiniteness of profinite residually solvable Lie algebras, thereby enriching the broader landscape of algebraic structures.

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For more of our work, please see [17 - 31]

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